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1995 J. Phys. A: Math. Gen. 28 L151

(http://iopscience.iop.org/0305-4470/28/5/002)

J. Phys. A: Math. Gen. **28** (1995) L151–L157. Printed in the UK

LETTER TO THE EDITOR

Multi-branch entrainment and multi-peaked order-functions in a phase model of limit-cycle oscillators with uniform all-to-all coupling

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Received 28 October 1994

Abstract. A peculiar type of mutual entrainment is found numerically in a phase model of a large population of limit-cycle oscillators with uniform all-to-all coupling strongly modulated by higher harmonics, in which more than one branch of mutually phase-locked oscillators exists, accompanied by a multi-peaked order function. For system size N, at least e^N different entrained states are likely to coexist with almost identical order functions. Also discussed is how to extend the order-function theory for dealing with such a phenomenon.

Large populations of coupled limit-cycle oscillators have been attracting much attention in diverse fields of science [1, 2]. From the point of view of dynamical systems theory, they provide an important category of large degrees of freedom systems that are now most intensively investigated in the area of nonlinear dynamics. On the other hand, assemblies of such oscillators can be viewed as a new type of co-operative systems exhibiting a phase-transition-like phenomenon, that is, macroscopic mutual entrainment in which a finite portion of element oscillators get phase-locked with one another, acquiring a common frequency. With a seemingly close analogy between that phenomenon and conventional phase transitions, it is an interesting subject to elucidate how far the analogy goes and where it breaks down. In theoretical studies of such systems, so-called phase models with uniform all-to-all coupling are often employed, which are of the form

$$d\theta_j/dt = \Omega_j + \frac{\epsilon}{N}\sum_{i=1}^{N} h(\theta_i - \theta_j) \qquad (j = 1, \ldots, N) \tag{1}$$

where $N(\gg 1)$ is the system size, θ_j the phase variable (divided by 2π) of the jth oscillator, Ω_j its intrinsic frequency, whose dispersal within the population is hereafter expressed by a density $f(\Omega)$, ϵ the control parameter, and $h(\theta) = h(\theta + 1)$ the coupling function. This type of model can be derived from underlying equations with an asymptotic analysis when coupling as well as when the frequency dispersion are weak, and have been investigated so far mostly for a particular case of $h(\theta) = \sin 2\pi\theta$, taken up first by Kuramoto two decades ago [3], which is because in that case, the model becomes not only simple but also analytically tractable (see, for example, [2, 4, 5]). Recently, however, a theory capable of dealing with generic coupling functions has been proposed [6], producing a number of fundamental results applicable to generic $h(\theta)$ and $f(\Omega)$ [6–8]. This theory is based on a concept of an *order function* (OF), $H(\theta)$, defined by

$$H(\theta) = -\sum_{k=-\infty}^{\infty} h_k Z_k e^{-2\pi i k\theta} \tag{2}$$

where $i \equiv \sqrt{-1}$ and h_k are Fourier components of $h(\theta)$: $h_k \equiv \int_0^1 d\theta\, h(\theta) e^{-2\pi i k\theta}$; Z_k are limits for $t \to \infty$ of $N^{-1}\sum_{j=1}^{N} e^{2\pi i k(\theta_j - \Omega_e t)}$, in which Ω_e is the frequency of entrainment. A self-consistent functional equation of $H(\theta)$ then follows, giving, in principle, all the information about the asymptotic state of the system as $t \to \infty$; indeed, its numerical solutions lead to excellent agreement with simulation results [6, 8]. A new solvable model has also been discovered on the basis of that equation [9]. In terms of this equation, the collective behaviour of oscillators corresponds to the bifurcation of a non-trivial solution from the trivial, which has been studied analytically in detail very recently, leading to, for example, a critical point formula to locate the onset of mutual entrainment [7] and generic scaling laws of order parameters and other quantities [8].

Yet the order-function theory may need to be further generalized since in the derivation of its equation, $H(\theta)$ is assumed to possess only one pair of minimum and maximums in a unit interval [6]; as can be seen from (2), this assumption should be valid for normal $h(\theta)$ such that its higher harmonic components are of moderate magnitude, but may not be correct otherwise. This paper demonstrates numerically that there is indeed such a case that the OF breaks the assumption by taking a multi-peaked shape; here, we focus on the case of $f(\Omega) = (0.4/\pi)(\Omega^2 + (0.4)^2)^{-1}$ and

$$h(\theta) = \sin 2\pi\theta + 0.2\cos 2\pi\theta - 0.3\sin 4\pi\theta + 0.6\cos 4\pi\theta + 0.7\sin 6\pi\theta - 0.4\cos 6\pi\theta \tag{3}$$

although similar results were obtained for some other cases as well. The coupling function, which already appeared in a previous paper [7], has a portrait as displayed in figure 1; the bumpy shape is due to the non-fundamental harmonics with fairly large magnitudes. Although it is not clear at the moment to what extent such a strongly modulated coupling can be relevant to real coupled-oscillator systems studied in laboratories or existing in nature, we will see below that it can generate a curious mode of mutual entrainment not known

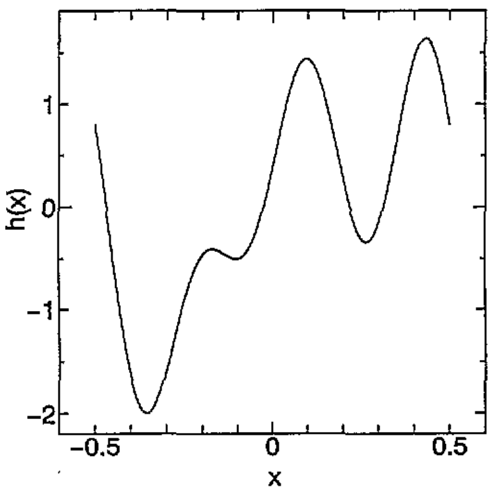


Figure 1. Portrait of (3).

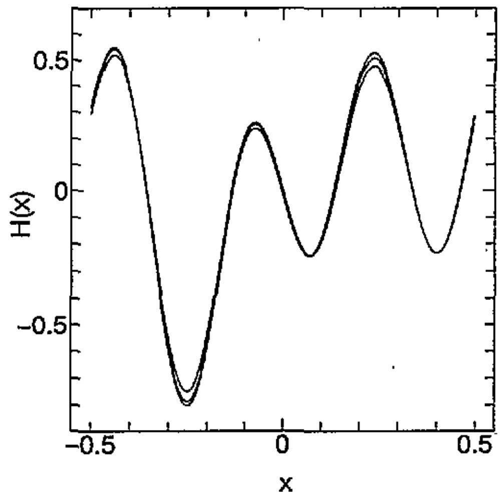


Figure 2. Portraits of order functions found by simulation for three different initial conditions. The method used to compute them is described in [8].

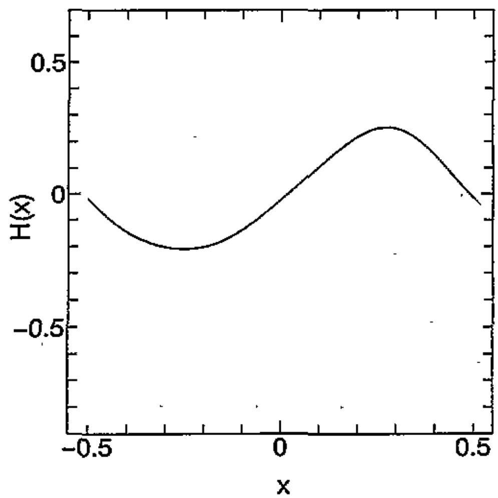


Figure 3. Order function obtained theoretically for the same ϵ as in figure 2.

before which is characterized by a multi-peaked OF. A sketch is also given of how to extend the theory in order to cope with such an 'abnormal' case.

Numerical integration of (1) was performed for $N = 4000$ by means of the Euler scheme with a time step of 0.01, starting from an initial condition prepared by a uniform random number generator. Some portraits of the OF obtained in this way for three different initial conditions are displayed in figure 2, where ϵ is 0.96. Remarkably, all of them have three pairs of minimum and maximums in the interval of length 1, looking qualitatively similar to the portrait of $h(\theta)$ presented above. Although those OFs are roughly of the same shape, it is evident that there exist slight differences. Figure 3 shows $H(\theta)$ obtained by numerically solving the theoretical functional equation for the same $h(\theta)$ $f(\Omega)$ and ϵ, which clearly disagrees with the results of numerical simulation. The reason for this would be either that mutual entrainment corresponding to the theoretical OF is unstable for the value of the control parameter or that the basin of attraction of such a state is too small to be easily detected. At any rate, the bifurcation theory [8] reveals that unlike a naive speculation made previously [7], the theoretical single-peaked OF is born via a normal bifurcation at $\epsilon = 0.8$, so that it is expected to be stable at least near its bifurcation point, though convincing

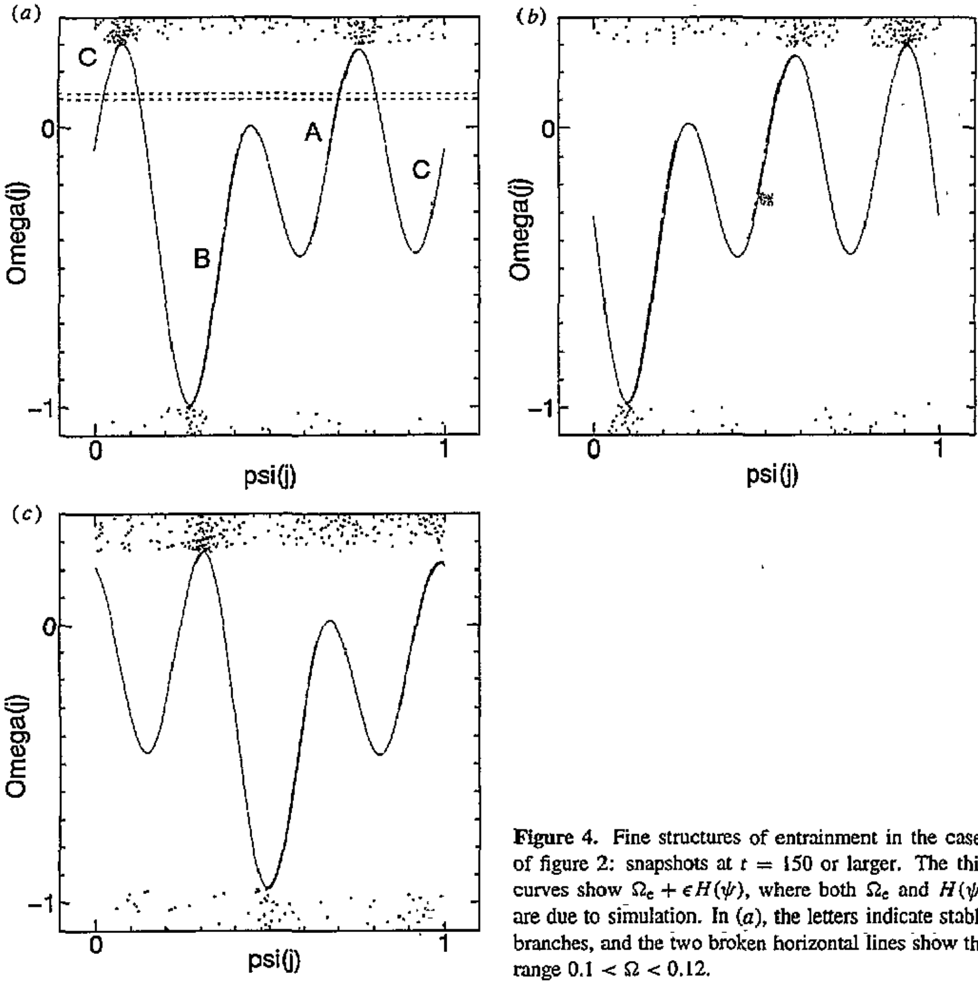


Figure 4. Fine structures of entrainment in the cases of figure 2: snapshots at $t = 150$ or larger. The thin curves show $\Omega_e + \epsilon H(\psi)$, where both Ω_e and $H(\psi)$ are due to simulation. In (a), the letters indicate stable branches, and the two broken horizontal lines show the range $0.1 < \Omega < 0.12$.

evidence has not yet been obtained. The multi-branch entrainment was found to occur for ϵ larger than about 0.6.

Now let us examine the nature of mutual entrainment with an OF multi-peaked as found above. For this sake, we may recall in what way the OF governs the asymptotic behaviour of the system. As discussed elsewhere [6], we have in the limit $t \to \infty$

$$d\psi_j/dt = \Delta_j - \epsilon H(\psi_j) \qquad (j = 1, \ldots, N) \tag{4}$$

where $\psi_j \equiv \theta_j - \Omega_e t$ and $\Delta_j \equiv \Omega_j - \Omega_e$. Let us write the minimum and maximum of $H(\theta)$ as $H_{\min}$ and $H_{\max}$, respectively. Then, it is easy to see that only oscillators with Δ_j falling in between $\epsilon H_{\min}$ and $\epsilon H_{\max}$ get mutually entrained, each having a constant residual phase given by $\psi_j = H^{-1}(\Delta_j/\epsilon)$, where it should be noted that $dH(\psi)/d\psi$ has to be positive for $\psi = \psi_j$ on account of stability. With these points in mind, we now go on to see what happens microscopically when the OF is multi-peaked as in figure 2. Figure 4 is devoted to this subject; for convenience, snapshots of Ω_j versus ψ_j are displayed, taken in the same runs as in figure 2. More than one branch of phase-locked oscillators is found to exist lying on curves showing $\Omega_e + \epsilon H(\psi)$, that is, curves on which every entrained oscillator has to settle for $t \to \infty$. It is therefore a multi-peaked OF that is responsible for

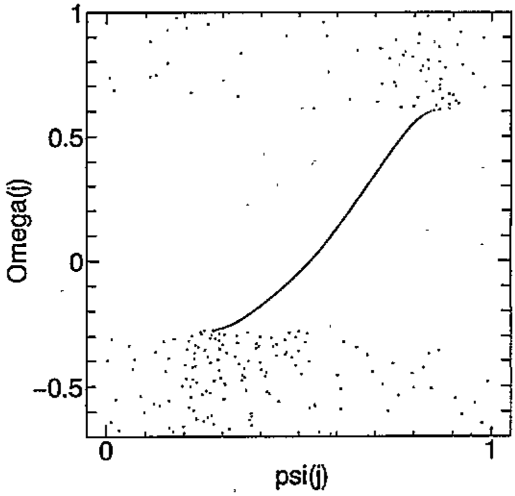
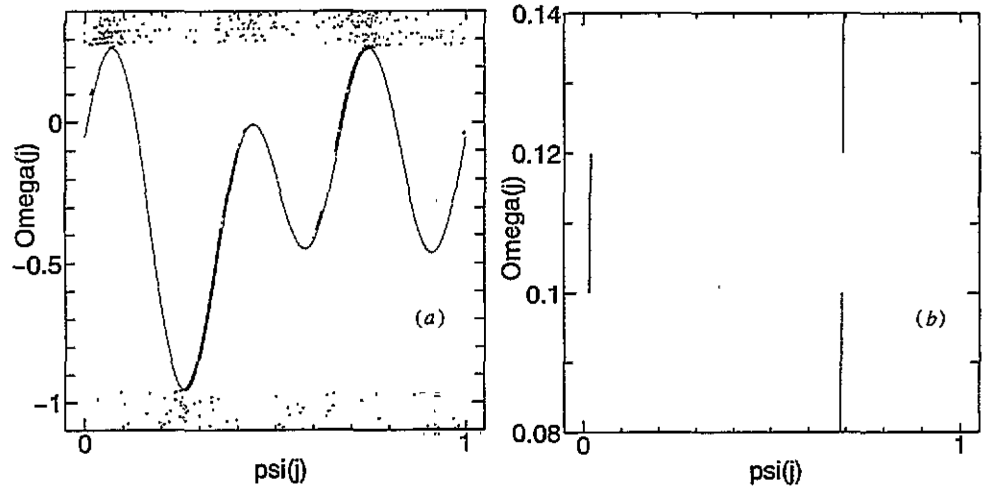


Figure 5. Example of single-branch entrainment, where $N = 1000$ and $\epsilon = 1.2$ with $h(\theta)$ equivalent to the $q = 0.5$ case discussed in [8] and $f(\Omega)$ as specified in the text.

the appearance of the multiple branches of entrained oscillators. For comparison, a typical example of the snapshot in the 'normal' case (for details, see the caption) is given in figure 5, where only one branch of entrainment is found; to the best of the author's knowledge, previously studied entrainments are all of this type, including that observed in Kuramoto's model. Here, it is important to note that the fine structure of entrainment depends on the initial condition. This problem will be considered in detail next.

Both figures 2 and 4 give evidence of the coexistence of different entrained states characterized by almost identical OFs. For a majority of entrained oscillators, there is more than one stable branch on which each of them could be placed; the main differences among those states seem to come from which branch is chosen by each of those oscillators. Let us consider, in a generalized context, how large the total number of such entrained states may be. Suppose that we have an entrained state such that H^{-1} has m stable branches in an interval, say, I and let N_e denote the total number of oscillators with Δ_j/ϵ belonging to I, where N_e is $O(N)$ since the entrainment is macroscopic. If N_e/N is finite but sufficiently less than unity, then every possible rearrangement of such oscillators by changing their branches would cause no substantial deformation of $H(\theta)$ (note the factor N^{-1} in (2)), thus self-consistency is maintained with a new stable entrained state possessing almost the same OF as the one before the rearrangement. We may therefore expect that at least $m^{N_e} \sim e^N$ different entrained states coexist with almost undistinguishable OFs. For $N \to \infty$, this implies a supermulti-basin structure for phase space. Creation of a new entrained state by changing the branches of some entrained oscillators is demonstrated in figure 6(a), which shows a snapshot taken some time after all the oscillators lying on part of branch A ($0.1 < \Omega_j < 0.12$) in figure 4(a) were transferred to another stable branch C without any other change made; a magnified view in figure 6(b) indicates that the transferred elements indeed remain on the new branch. Figure 6(c) displays OFs before and after the rearrangement; nothing changes drastically as expected.

Coexistence of a large number of attractors is discussed in the literature for systems consisting of identical elements [10], in which case permutation symmetry makes the occurrence of such a phenomenon almost trivial, and the estimate of the number of attractors is straightforward. What most distinguishes the present case from this is the fact that here the elements are not identical because of their characteristic frequencies. In such a case, the estimate mentioned above is, in general, difficult even at a semi-quantitative level,



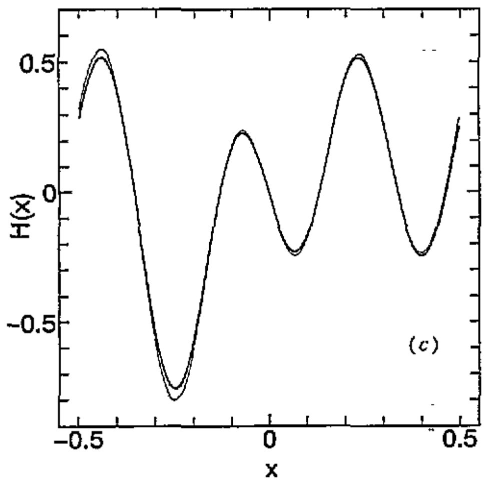


Figure 6. New entrained state created from the one displayed in figure 4(a) by a rearrangement of entrained oscillators in the range shown in figure 4(a): (a) snapshot taken $\Delta t = 180$ after the rearrangement; (b) its magnification; (c) new OF in comparison with the old.

but here the OF has made it possible to some extent. Without the frequency dispersal, model (1) is known to exhibit a phenomenon called 'clustering' when $h(\theta)$ includes higher harmonic components [11]. Although irrelevant to mutual entrainment *per se*, it may have a connection with the above phenomenon.

The self-consistent equation of $H(\theta)$ derived previously [6] cannot be employed for the purpose of analysing multi-branch entrainments. Is it then possible to generalize it for that purpose? Actually, this is seemingly quite easy: suppose that $H(\theta)$ is shaped as shown in figure 7, and write the density of oscillators on the lth stable branch as $\rho_l(\Omega)$; then, by definition, $\rho_l(\Omega) = 0$ for $\Omega < \Omega_e + \epsilon H(X_l)$ and $\Omega > \Omega_e + \epsilon H(Y_l)$ and also $\sum_{l=1}^{m} \rho_l(\Omega) = f(\Omega)$ for $\Omega_e + \epsilon H_{\min} < \Omega < \Omega_e + \epsilon H_{\max}$. Following [6], we can find

$$H(\theta) = -\epsilon \sum_{l=1}^{m} \int_{X_l}^{Y_l} \mathrm{d}\psi\, \rho_l(\Omega_e + \epsilon H(\psi)) H'(\psi) h(\psi - \theta)$$

$$- \int_{\Delta > \epsilon H_{\max}, \Delta < \epsilon H_{\min}} \mathrm{d}\Delta f(\Omega_e + \Delta) C(\Delta) \int_0^1 \mathrm{d}\psi \frac{h(\psi - \theta)}{\Delta - \epsilon H(\psi)} \qquad (5)$$

where $C(\Delta) \equiv 1/\int_0^1 \mathrm{d}\psi/(\Delta - \epsilon H(\psi))$. This equation does not appear quite tractable

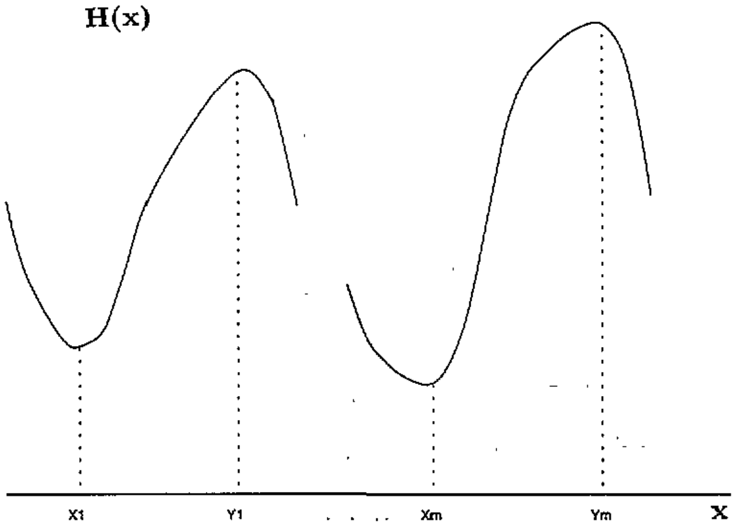


Figure 7. Fully generalized form of $H(\theta)$ with X_i and Y_i locating its local minima and maxima, respectively.

because of the entry of $\rho_l(\Omega)$, but its analysis may reveal, for example, conditions on $h(\theta)$ (and $f(\Omega)$) for the OF to become multi-peaked. Some attempts will be made elsewhere.

To summarize, a new mode of mutual entrainment, multi-branch entrainment, in globally coupled phase oscillators has been reported and its nature has been discussed in the light of the order function, whose crucial role should be noticed in understanding both the mechanism of that entrainment and the associated supermulti-basin structure in phase space. This phenomenon originates from strongly modulated coupling which may possibly appear in assemblies of highly nonlinear oscillators; experimental efforts towards discovering multi-branch entrainment may be rewarding for such systems.

This work is supported by a Grant-in-Aid from the Ministry of Education, Science, and Culture (no 06835020).

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